Redheffer Notes 7

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Let μ be the Mobius function and M be the Mertens function. Consider the following $n \times n$ matrices:

$$
\begin{array}{rcc}
[A]_{i,j} & := & \begin{cases} 1 \text{ if } j = ik, k \in \mathbb{N} \\ 0 \text{ else} \end{cases} \\
[M]_{i,j} & := & \begin{cases} \mu(k) \text{ if } j = ik, k \in \mathbb{N} \\ 0 \text{ else} \end{cases} \\
[F]_{i,j} & := & \lfloor \frac{j}{i} \rfloor \\
[E]_{i,j} & := & M(\lfloor \frac{j}{i} \rfloor) \\
[U]_{i,j} & := & \begin{cases} 1 \text{ if } i \leq j \\ 0 \text{ else} \end{cases}
\end{array} \tag{1}
$$

Theorem 1. *We show the following matrix identities:*

$$
\begin{array}{rcl}
AU & = & F \\
MU & = & E \\
MF & = & U \\
AE & = & U
\end{array} \tag{2}
$$

Proof. Note that $A(i,:)$ has a 1 every i columns; therefore, $A(i,:) \begin{pmatrix} 1 & \\ 1 & \\ \vdots & \}\text{j times} \\ 1 & \\ 0 & \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} = A(i,:)U(:,j)$ is the number of times i goes into m without remainder, or $\lfloor \frac{j}{i} \rfloor$. This is the first identity.

The second identity is similarly straightforward as $[MU]_{i,j} = \Sigma_{k=1}^{j} [M]_{i,k} = \Sigma_{k=1}^{\lfloor \frac{j}{i} \rfloor} \mu(k) =: M(\lfloor \frac{j}{i} \rfloor)$.

The third identity is a direct consequence of Mobius inversion as $[MF]_{i,j} = \Sigma_{k=1}^{n} [M]_{i,k} \lfloor \frac{j}{i} \rfloor = \Sigma_{l=1}^{\lfloor \frac{j}{i} \rfloor} \mu(l) \lfloor \frac{j}{il} \rfloor$ which since as in the first identity $\lfloor \frac{j}{i} \rfloor = \Sigma_{l=1}^{\frac{j}{i}} 1$ then by a generalized Mobius inversion this $= 1$.

The fourth identity is a corollary of the previous identities, since:

(all matrices defined are upper triangular with all nonzeros along the diagonal and are therefore invertible)

$$
\begin{array}{rcl}
MF & = & U \\
MUU^{-1}F & = & U \\
EU^{-1} & = & UF^{-1} \\
EU^{-1} & = & A^{-1} \\
AE & = & U
\end{array} \tag{3}
$$

□

Now let R is the Redheffer matrix and $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_{-1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. What follows is a matrix-identities-only proof of Redheffer.

Theorem 2.

$$det(R) = \Sigma_{k=1}^{n} \mu(k) \tag{4}$$

Proof.

$$det(R) \quad = \quad det(A + e_{-1}e_1^T) \tag{5}$$

By the Matrix-determinant lemma:

$$
\begin{array}{rc}
= & (1 + e_1^T A^{-1} e_{-1}) det(A) \\
= & (1 + e_1^T A^{-1} e_{-1})
\end{array} \tag{6}
$$

By the previous theorem:

$$
\begin{array}{rc}
= & (1 + e_1^T EU^{-1} e_{-1}) \\
= & (1 + e_1^T M e_{-1}) \\
= & \Sigma_{k=1}^{n} \mu(k)
\end{array} \tag{7}
$$

□

Note that this proof combined with the theorems above give rise to a number of different, equivalent matrix determinant problems.

I have nowhere else to put this, but–

Remark 2.1. Let

$$[D]_{i,j} \quad := \quad \frac{j}{i}, j \geq i \tag{8}$$

Then $D^{-1} = \begin{pmatrix} 1 & -\frac{2}{1} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{3}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{n}{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$.

I have not been able to find any useful facts using this in conjunction with F.